Chapter 2

Definitions and Concepts from General Relativity

The primary purpose of this book is to show how it is possible for the fundamental particles and the forces of nature to be conceptually explained using only 4 dimensional spacetime. This is nothing less than a new model of the universe. Presentation of this model would be an impossibly large task for a single person if every aspect of the model needs to be analyzed with the detail that might be expected from a technical paper covering a specific aspect of a mature subject. Therefore, the concepts will be introduced using simple equations that involve approximations and ignore dimensionless constants such as 2π or $\frac{1}{2}$. These simplifications permit the key concepts of this very large subject to be explained. Later, others can analyze and expand upon this large subject in more detail.

We will start by looking at the gravitational effects on spacetime in the limiting case of weak gravity. Much of the analysis to follow in subsequent chapters will deal with the gravity exhibited by a single fundamental particle such as an electron or a quark. Working with single particles or the interaction between two fundamental particles allows the proposed structure of such particles to be connected to the forces that are exhibited by these particles. At the same time, the extremely weak fields permit simplifications in the analysis.

In the following discussion, a distinction will be made between the words “length” and “distance”. Normally, these words are similar, but we will make the following distinction. Length is a spatial measurement standard. This is not just a standardized size such as a meter or inch, but it also can include a qualification such as proper length or coordinate length. A unit of length can be defined either by a ruler or by the speed of light and a time interval. The concept of distance as used here is best illustrated by the phrase “the distance between two points”. A distance can be quantified as a specific number of length units.

This chapter starts off with a discussion of the Schwarzschild solution to the Einstein field equation and physical examples of the effect of gravity on spacetime. This will seem elementary to many scientists, but new terminology and physical interpretations are introduced. Understanding this terminology and perspective is a requirement for subsequent chapters.

Schwarzschild Solution of the Field Equation: Einstein’s field equation has an exact solution for the simplified case of a static, nonrotating and uncharged spherically symmetric mass distribution in an empty universe. This mass distribution with total mass m is located at the origin of a spherical coordinate system. The standard (nonisotropic) Schwarzschild solution in this case takes the form:

$dS^2 = c^2d\tau^2 = (1/\Gamma^2)c^2dt^2 - \Gamma^2dR^2 - R^2d\Omega^2$

$\Gamma \equiv dt/d\tau = \frac{1}{\sqrt{1-\left(\frac{2Gm}{c^2R}\right)}}$ (explained below)

$dS = cd\tau$ The invariant quantity dS is the length of the world line between two events in 4 dimensional spacetime.
R = circumferential radius (circumference/2π) [explained below]
Ω = a solid angle in a spherical coordinate system ($d\Omega^2 = d\theta^2 + sin^2\theta\, d\Phi^2$)
c = the speed of light constant of nature
t = coordinate time (time infinitely far from the mass - effectively zero gravity)
τ = proper time – time interval on a local clock in gravity

Circumferential radius: Gravity warps the space around mass, so that the space around the test mass has a non-Euclidian geometry. The circumference of a circle around the mass does not equal 2π times the radial distance to the center of mass. To accommodate this warped space, the Schwarzschild equation uses a special definition of distance to specify the coordinate distance in the radial direction. Names like "R-coordinate" and "reduced circumference" are sometimes used to describe this radial coordinate that cannot be measured with a meter stick or a pulse of light. The name that will be used here is "circumferential radius" and designated with the symbol R. This is a distance that is calculated by measuring the circumference of a circle that surrounds a mass, and then dividing this circumference by 2π. If we measure the radius using a hypothetical meter stick or tape measure, then the "proper" radial distance will be designated by r. We can see from the Schwarzschild metric that if we set $dt = 0$, $dS = cd\tau = dr$ and $d\Omega = 0$ then:

$dr = \Gamma\, dR$

Gravitational Gamma Γ: The metric has been written in terms of the quantity Γ which this book will refer to as the "gravitational gamma Γ". The basic definition of Γ is:

$$\Gamma \equiv dt/d\tau = \frac{1}{\sqrt{1+\left(\frac{2\varphi}{c^2}\right)}} = \frac{1}{\sqrt{1-\left(\frac{2Gm}{c^2R}\right)}} = \frac{1}{\sqrt{g_{oo}}}$$

where Γ = gravitational gamma and $\varphi = -Gm/R$ gravitational potential
$\Gamma = dt/d\tau$ in the static case when $dR = 0$ and $d\Omega = 0$.
g_{oo} is a metric coefficient commonly used in general relativity

The symbol of upper case gamma Γ was chosen because this equation can also be written as follows:

$\Gamma = \frac{1}{\sqrt{1-\left(\frac{V_e^2}{c^2}\right)}}$ where $V_e \equiv \sqrt{\frac{2Gm}{R}}$ = escape velocity

The similarity between Γ and γ of special relativity is obvious since: $\gamma = \frac{1}{\sqrt{1-\left(\frac{V^2}{c^2}\right)}}$

This analogy between the escape velocity in general relativity and the relative velocity in special relativity extends further in the weak gravity approximation. The time dilation due to gravity approximately equals the time dilation due to relative motion when the relative motion is equal to the gravitational escape velocity (weak gravity).

The Schwarzschild universe with only one mass has effectively "zero gravity" at any location infinitely far from the mass. At such a location $\Gamma = 1$. The opposite extreme of the maximum possible value of Γ is the event horizon of a black hole where $\Gamma = \infty$. If the earth was in an empty universe, then the surface of the earth would have a gravitational gamma of: $\Gamma \approx 1 + 7 \times 10^{-10}$. It is important to remember that Γ is always larger than 1 when gravity is present.

A stationary clock infinitely far away from the mass in Schwarzschild's universe is designated the "coordinate clock" with a rate of time designated dt. In the same stationary frame of reference, the local rate of time in gravity (near the mass) is designated $d\tau$. The relationship is: $\Gamma = dt/d\tau$

There is a useful approximation of Γ that is valid for weak gravity.

$\Gamma \approx 1 + \frac{Gm}{c^2 R}$ weak gravity approximation of Γ

Gravitational Magnitude *β*: The gravitational gamma Γ has a range of possible values that extends from 1 to infinity. There is another related concept where the strength of the gravitational effect on spacetime ranges from 0 to 1 where 0 is a location in zero gravity and 1 is the event horizon of a black hole. This dimensionless number will be called the "gravitational magnitude β" and is defined as:

$\beta \equiv 1 - \frac{d\tau}{dt} = 1 - \sqrt{1 - \frac{2Gm}{c^2 R}} = 1 - \frac{1}{\Gamma}$ $\beta =$ gravitational magnitude

In weak gravity the following approximation is accurate:

$\beta \approx \frac{Gm}{c^2 r}$ and $\Gamma \approx 1 + \beta$ weak gravity approximations

For example, the earth's gravity, in the absence of any other gravity is $\beta \approx 7 \times 10^{-10}$. The sun's surface has $\beta \approx 2 \times 10^{-6}$ and the surface of a hypothetical neutron star with escape velocity equal to half the speed of light would have $\beta \approx 0.13$.

In common usage, the strength of a gravitational field is normally associated with the acceleration of gravity. However, the acceleration of gravity depends on the gradient of gravitational potential. In contrast, the gravitational magnitude β and the gravitational gamma Γ are measurements of the effect of gravity on time and distance without regard for the gravitational gradient. For example, there is an elevation in Neptune's atmosphere where Neptune has approximately the same gravitational acceleration as the earth. However, Neptune has roughly 16 times the earth's mass and roughly 4 times the earth's radius. This means that Neptune's gravitational magnitude β is roughly 4 times larger than earth's at locations where the gravitational acceleration is about the same.

The gravitational magnitude approximation $\beta \approx Gm/c^2r$ will be used frequently with weak gravity. For example, this approximation is accurate to better than one part in 10^{36} for examples that will be presented later involving the gravity of a single fundamental particle at an important radial distance. Therefore, this approximation will be considered exact when dealing with fundamental particles. Note that this approximation includes the substitution of proper radial distance r for the circumferential radius R. $(r \approx R)$. Using the approximation $\beta \approx Gm/c^2r$ we also obtain the following equalities for β in weak gravity:

$\beta \approx \frac{-\varphi}{c^2} \approx \frac{R_s}{r}$ $\qquad$ $\varphi = -Gm/R$ and $R_s = Gm/c^2$ = Schwarzschild radius* (see note below)

$\beta \approx \frac{dt - d\tau}{dt}$ $\qquad$ the rate of time approximation for weak gravity

All of these approximations will be considered exact when dealing with the extremely weak gravity of single fundamental particles in subsequent chapters.

- Note: The Schwarzschild radius of a nonrotating black hole is $r_s = 2Gm/c^2$ and the Schwarzschild radius of maximally rotating black hole, such as a photon black hole, is $R_s = Gm/c^2$. This book will often ignore numerical constants near 1, but there is another reason for using $R_s = Gm/c^2$ for particles. The particle model proposed later in this book has energy rotating at the speed of light which would have gravitational effects scale with R_s rather than r_s.

Zero Gravity: Schwarzschild assumed an empty universe with only a single mass. Such a universe approaches zero gravity as the distance from the mass approaches infinity. However, is there anywhere in our observable universe that can truly be designated as a zero gravity location? There are vast volumes with virtually no gravitational acceleration compared to the cosmic microwave background. However, this is not the same as saying that these volumes have a gravitational gamma of $\Gamma = 1$ (a hypothetical empty universe). Everywhere in the real universe there is gravitational influence from all the mass/energy in the observable universe. Later, in the chapters on cosmology, an attempt is made to estimate the background (omni-directional) gravitational gamma of our observable universe compared to a hypothetical empty universe. While the presence of a uniform background Γ for the universe has implications for cosmology, we can only measure differences in Γ. There is ample evidence that general relativity works well by simply ignoring the uniform background Γ of the universe. Effectively we are assigning $\Gamma = 1$ to the background gravitational gamma of the universe and proportionally scaling from this assumption.

Therefore, a distant location which we designate as having $\beta = 0$ or $\Gamma = 1$ will be referred to as a "zero gravity location" or simply "zero gravity". The term "zero gravity" in common usage usually implies the absence of gravitational acceleration as might be experienced in free fall. However, in this book "zero gravity" literally means that we are using the Schwarzschild model of a distant location which has been assigned coordinate values of $\beta = 0$ and $\Gamma = 1$. The rate of time at this coordinate location will be designated dt and called "coordinate rate of time". A clock at this location will be designated as the "coordinate clock".

Gravitational Effect on the Rate of Time: The equation $dt = \Gamma d\tau$ is perhaps the most important and easiest to interpret result of the Schwarzschild equation. It says that the rate of time depends on the gravitational gamma Γ. This equation has been proven correct by numerous experiments. Today the atomic clocks in GPS satellites are routinely calibrated to account for the different rate of time between the lower gamma at the GPS satellite elevation and the higher gamma at the Earth's surface. Without accounting for this gravitational relativistic effect, the GPS network would accumulate errors and cease to function accurately after about one day. (There is also time dilation caused by the relative motion of the satellite. This is a much smaller correction than the gravitational effect and in the opposite direction.)

The difference in the rate of time with respect to radial distance in gravity will be called the "gravitational rate of time gradient". The gravitational rate of time gradient is not a tidal effect. An accelerating frame of reference has no tidal effects, yet it exhibits a rate of time gradient. Our objective is to provide an equation that relates the acceleration of gravity "g" to the rate of time gradient and utilizes proper length in the expression of the rate of time gradient. For example, inside a closed room it is possible to measure the gravitational acceleration. If we cannot measure any tidal effects, there is no information about the mass and distance of the object producing the gravity. Is it possible to determine the local rate of time gradient (expressed using proper length and proper time) from just the gravitational acceleration?

It has been shown[1] that a uniform gravitational field with proper acceleration g (measured locally), has the following relationship between redshift and gravitational acceleration:

[1] Edward A. Desloge, "The Gravitational Redshift in a Uniform Field" Am. J. Phys. 58 (9), 856-858 (1990)

$\upsilon/\upsilon_o = 1 - gh/c^2$ where:

υ_o = frequency as measured at the source location with rate of time $d\tau_o$
υ = frequency as measured at the detector location with rate of time $d\tau$
h = vertical distance using proper length between the source and detector
g = acceleration of gravity

Reference [1] shows that this equation is <u>exact</u> if the following qualifications are placed on the above definitions. These qualifications are: 1) the source location (subscript o) should be at a lower elevation than the detector location. 2) the separation distance h should be the proper length as measured by a time of flight measurement (radar length) measured from the source location. A slightly different radar length would be obtained if this distance was measured from the detector elevation or measured with a ruler. However, in the limit of a gradient (infinitely small h), this discrepancy disappears. Therefore with these qualifications:
$\upsilon/\upsilon_o = 1 - gh/c^2$ is exact. It should be noted that the height difference h is a proper distance (radar length measured from the source) and not circumferential radius.

As will be show in the next chapter, the gravitational redshift is really caused by a difference in the rate of time at different elevations. There is no accumulation of wavelengths, so $\upsilon_o = 1/d\tau_o$ and $\upsilon = 1/d\tau$. After making these substitutions, this equation becomes:

$$g = c^2 \frac{d\tau - d\tau_o}{d\tau\, dh}$$

This is also an exact equation if the above qualifications are observed. Here the ratio $(d\tau - d\tau_o)/d\tau dh$ will be referred to as the gradient in the rate of time. There are two points to be noticed. First, the gradient in the rate if time is able to be determined from the acceleration of gravity with no knowledge about the mass or distance of the body producing the gravity. For example, a gravitational acceleration of $g = 1$ m/s^2 is produced by a rate of time gradient of 1.113×10^{-17} seconds/second per meter. The earth's gravitational acceleration of 9.8 m/s^2 near the earth's surface is caused by a rate of time gradient of about 10^{-16} seconds/second per meter of elevation difference in the earth's gravity. The most accurate clock presently available (2015) is an ^{87}Sr optical lattice clock[2] which has an accuracy of 2.1×10^{-18}. A clock with this accuracy has a resolution comparable to 2 cm elevation change in the earth's gravity.

The second important point is that the rate of time gradient is a function of proper length in the radial direction. Even though $dt/d\tau$ is a function of circumferential radius, the relationship between rate of time gradient and gravitational acceleration is not a function of circumferential radius. This fact will become important in the next chapter when we examine how nature keeps the laws of physics constant when there is an elevation change. The connection between

[2] http://www.nature.com/ncomms/2015/150421/ncomms7896/full/ncomms7896.html

gravitational acceleration and rate of time gradient will also be an important consideration when we examine the cosmological model of the universe (chapters 13 & 14).

Previously, we defined the concept of "gravitational magnitude β" as: $\beta = 1 - d\tau/dt$. It is also possible to relate the acceleration of gravity to the gradient in the gravitational magnitude.

$$g = c^2\left(\frac{d\beta}{d\hbar}\right)$$

Inertial Frame of Reference: The concept that gravity can be simulated by an accelerating frame of reference sometimes leads to the erroneous interpretation that an inertial frame of reference eliminates all effects of gravity. Being in free fall eliminates the acceleration of gravity, but the gravitational effect on the rate of time and the spatial effects of the gravitational field remain. Another way of saying this is that the effects of the gravitational gamma Γ on spacetime are still present, even if a mass is in an inertial frame of reference. A clock in free fall still experiences the local gravitational time dilation.

A rigorous analysis from general relativity confirms this point, but two examples will be given to also illustrate the concept. Suppose that there was a hollow cavity at the center of the earth. A clock in this cavity would experience no gravitational acceleration and would be in an inertial frame of reference. The gravitational magnitude β in this cavity is about 50% larger than the gravitational magnitude on the surface of the earth ($\sim 10.5 \times 10^{-10}$ compared to 7×10^{-10}). For example, ignoring air friction, the escape velocity starting from this cavity is higher than starting from the surface of the earth. The clock in the cavity has a slower rate of time than a clock on the surface. The inertial frame of reference does not eliminate the other gravitational effects on the rate of time and the gravitational effect on volume.

A second example is interesting and illustrates a slightly different point. The Andromeda galaxy is 2.5 million light years ($\sim 2.4 \times 10^{22}$ m) away from Earth and has an estimated mass of about 2.4×10^{42} kg (including dark matter). The gravitational acceleration exerted by this galaxy at the distance of the Earth is only about 2.8×10^{-13} m/s^2. To put this minute acceleration in perspective, a 10,000 kg spacecraft would accelerate at about this rate from the "thrust" of the light leaving a 1 watt flashlight. In spite of the minute gravitational acceleration, the distant presence of Andromeda slows down the rate of time on the surface of the earth about 100 times more than the Earth's own gravity. This is possible because the gravitational magnitude ($\beta \approx Gm/c^2r$ for weak gravity) decreases at a rate of $1/r$ while the gravitational acceleration decreases with $1/r^2$. At the earth's surface, Andromeda's gravitational magnitude is about:

$\beta \approx Gm/c^2r = (G/c^2)\,(2.4 \times 10^{42}\text{ kg}/2.4 \times 10^{22}\text{ m}) \approx 7 \times 10^{-8}$ Andromeda's β at earth

Since the earth's gravity produces $\beta \approx 7 \times 10^{-10}$ at the surface, Andromeda's effect on the rate of time at the earth's surface is about 100 times greater than the effect of the earth's gravity. It does not matter whether a clock is in free fall relative to Andromeda or whether the clock is stationary relative to Andromeda and experiences the minute gravitational acceleration. In both cases the gravitational effect on time and volume exist. This example also hints that mass/energy in other parts of the universe can have a substantial cumulative effect on our local rate of time and our local volume. This concept will be developed later in the chapters dealing with cosmology.

Schwarzschild Coordinate System: The standard Schwarzschild solution uses coordinates that simplify gravitational calculations. This spherical coordinate system uses circumferential radius R as coordinate length in the radial direction and uses circumferential radius times an angle Ω for the tangential direction. While there is no distinction in proper length for the radial and tangential directions, we will temporarily make a distinction by designating proper length in the radial direction as L_R and designating proper length in the tangential direction as L_T. This distinction does not exist in reality since: $cd\tau = dL = dL_T = dL_R$. However, using these designations, the relationship between proper length and Schwarzschild's coordinate length is:

$dL_R = \Gamma\, dR$ radial length L_R conversion to Schwarzschild radial coordinate R
$dL_T = R\, d\Omega$ tangential length L_T to Schwarzschild tangential coordinate length

The equation $dL_R = \Gamma dR$ is obtained by setting $dt = 0$. If we are using the proper distance between two points as measured by a ruler, or the calculated circumferential radius, then this zero time assumption is justified.

Next we will calculate the coordinate speed of light $\mathcal{C}$ for the radial and tangential directions by starting with the standard Schwarzschild metric:

$dS^2 = (1/\Gamma^2)c^2dt^2 - \Gamma^2 dR^2 - R^2 d\Omega^2$ for light set $dS^2 = 0$,
$(1/\Gamma^2)\, c^2\, dt^2 = \Gamma^2 dR^2 + R^2 d\Omega^2$

$$c^2 = \Gamma^4 \frac{dR^2}{dt^2} + \Gamma^2 \frac{R^2 d\Omega^2}{dt^2}$$

If we separate this coordinate speed of light into its radial component ($\mathcal{C}_R$) and its tangential component ($\mathcal{C}_T$), we obtain:

$\mathcal{C}_R = dR/dt = c/\Gamma^2$ $\mathcal{C}_R$ = coordinate speed of light in the radial direction ($d\Omega = 0$)
$\mathcal{C}_T = Rd\Omega/dt = c/\Gamma$ $\mathcal{C}_T$ = coordinate speed of light in the tangential direction ($dR = 0$)

This apparent difference in the coordinate speed of light for the radial and tangential directions is not expressing a physically measurable difference in the proper speed of light. The difference follows from the standard (nonisotropic form) of the Schwarzschild metric. If we choose the isotropic form of the Schwarzschild metric the difference will disappear and $\mathcal{C}_R = \mathcal{C}_T \approx c/\Gamma^2$.

However, the isotropic form has its own set of complexities, so we will be using the standard Schwarzschild metric.

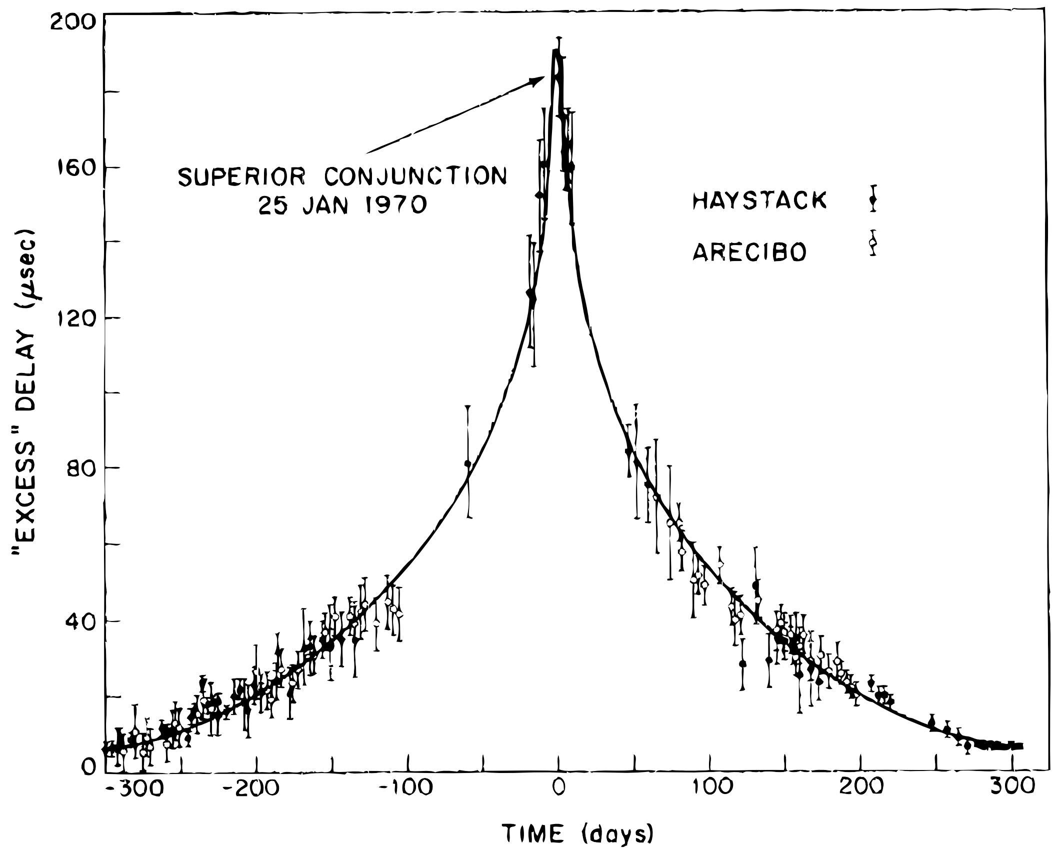

FIGURE 2-1 This is Irwin Shapiro's figure showing the relativistic time delay caused by the sun's gravity on the round trip time for radar to travel from the earth to Venus and back. The x axis is time in days before and after superior conjunction of Venus passing behind the sun.

The Shapiro Experiment: Next, we are going to switch to a discussion about the gravitational effect on proper length, proper volume and the coordinate speed of light. In 1964, Irwin Shapiro proposed an experiment to measure the relativistic distortion of spacetime caused by the Sun's gravity. This non-Newtonian time delay is obtained from the Schwarzschild solution to Einstein's field equation. The Sun is a good approximation of an isolated mass addressed by the Schwarzschild solution. The implication is that gravity affects spacetime so that it takes more time for light to make the round trip between two points in space when the mass (gravity) is present than when the mass (gravity) is absent. Shapiro and his colleagues used radar to track the planet Venus for about two years as Venus and the Earth orbited the Sun. During this time, Venus passed behind the Sun as seen from the Earth (nearly superior conjunction). The orbits of Venus and the Earth are known accurately, so it was possible to measure the additional time delay in the round trip time from the earth to Venus and back. The effect of the sun's gravity on this round trip time could be calculated from multiple measurements made over the two year time period. Figure 2-1 shows Shapiro's graph of the excess time delay over the two year period. The peak delay at superior conjunction was 190 μs on a half hour round trip transit time.

Variations of this experiment have been repeated numerous times in the normal course of the space program. Spacecraft on their way to the outer planets often start with an orbital path that at some point results in nearly superior conjunction relative to the Earth. The most accurate measurement to date was with the Cassini spacecraft. It was equipped with transponders at two different radar frequencies, therefore it was possible to determine and remove the effect of the Sun's corona on the time delay. The result was an agreement with the time delay predicted by general relativity accurate to 1 part in 50,000. With this type of agreement, it would seem as if there are no remaining mysteries about this effect and the physical interpretation should be obvious.

How exactly do length, time and the speed of light combine to produce the observed time delay in the Shapiro effect? For simplicity, we would like to look at the time delay associated with a radar beam traveling only in the radial direction. In the limit, we can imagine reflecting a radar beam off the surface of the Sun. This path would be purely radial. To make a measurement we need to have a round trip, but for simplicity of discussion, we will talk about the time delay for a one way trip. For light $dS = 0$, so the metric equation gives us that for light moving in the radial direction:

$$\left(\frac{1}{\Gamma}\right) cdt = \Gamma dR \rightarrow dt = \left(\frac{\Gamma^2}{c}\right) dR$$

We can compute the time Δt it takes to move between two different radii: r_1 and r_2 (where $r_2 > r_1$).

$$c\Delta t = \int c\ dt\ =\ \int_{r1}^{r2} \Gamma^2\ dR$$

$$c\Delta t = \left(\frac{2Gm}{c^2}\right) ln\left(\frac{r_2}{r_1}\right) + \left(\frac{2Gm}{c^2}\right) ln\left(\frac{\Gamma_1^2}{\Gamma_2^2}\right) \qquad \text{weak gravity:}\quad ln\left(\frac{\Gamma_1^2}{\Gamma_2^2}\right) \approx 0$$

$$\Delta t \approx \left(\frac{2Gm}{c^3}\right) ln\left(\frac{r_2}{r_1}\right)$$

Substituting the Sun's mass, radius and distance gives $\Delta t \approx 50$ μs. Therefore, in addition to a non-relativistic time delay (about 8 minutes), the one way relativistic time delay would be about an additional 50 μs. The 190 μs delay observed by Shapiro is roughly 4 times the one way 50 μs delay from the earth to the sun because of the additional leg to Venus and then the round trip doubling of the time.

Normally, on Earth we would interpret a 50 μs delay in a radar beam as indicating an additional distance of about 15 km. How much does the sun's gravity distort space and increase the radial distance between the earth and the sun's surface compared to the distance that would exist if we had Euclidian flat space? The additional non Euclidian path length will be designated (ΔL). Starting from: $dL_R = \Gamma\ dR$

$\Delta L = \int_{r1}^{r2} \Gamma\ dR$

$\Delta L \approx \left(\frac{Gm}{c^2}\right) ln\left(\frac{r_2}{r_1}\right)$ set $r_2 = 1.5 \times 10^{11}$ m, $r_1 = 7 \times 10^8$ m and $m = 2 \times 10^{30}$ kg

$\Delta L \approx 7.5$ km non-Euclidian additional proper distance between the Earth and Sun

Suppose that it was possible to stretch a tape measure from the earth to the surface of the sun. The distance measured by the tape measure (proper distance) would be about 7.5 km greater than a distance obtained from an assumption of flat space and a Euclidian geometry calculation. The use of a tape measure means that we are using proper length as a standard.

Gravity Increases Volume: If we use proper length as our standard of length rather than circumferential radius, then we must adopt the perspective that gravity increases the volume of the universe. However, an interpretation based on proper volume is often ignored since the use of circumferential radius as coordinate length eliminates this volume change caused by gravity.

In the Shapiro experiment, we calculated that there was a non-Euclidian increase in distance between the earth and the sun of 7.5 km. Suppose that we imagine a spherical shell with a radius equal to the average radius of the earth's orbit. This is a radial distance equal to one astronomical unit (AU $\approx 1.5 \times 10^{11}$ m). The sun's mass is $\approx 2 \times 10^{30}$ kg and the sun's Schwarzschild radius is $r_s \approx 2950$ meters. What is the change in volume (ΔV) inside this spherical shell if we compare the Euclidian volume of the shell (V_o) and the non-Euclidian volume of the shell (V) when the Sun is at the center of the shell?

$V = \int dV = \int 4\pi R^2\ \Gamma\ dR$

After integration, the difference $\Delta V = V - V_o$ is approximately:

$\Delta V \approx (5\pi/3) r_s (r_2^2 - r_1^2)$ set $r_2 = 1.5 \times 10^{11}$ m, $r_1 = 7 \times 10^8$ m and $r_s = 2950$ meters

$\Delta V \approx 3.46 \times 10^{26}$ m^3 non Euclidian volume increase

To put this non Euclidian volume increase in perspective, the sun's gravity has increased the proper volume within a radius of 1 AU by about 3.5×10^{26} m^3 which is more than 300,000 times larger than the volume of the earth (earth's volume is $\approx 1.08 \times 10^{21}$ m^3). Stated another way, the volume increase is about 20% smaller than the volume obtained by multiplying the non-Euclidian radial length increase (~ 7,500 m) times the surface area of the spherical shell with a radius of 1 AU. Obviously this non Euclidian volume increase would be much larger if we had chosen a larger shell radius (for example, the size of the observable universe). The implications of this will be explored in the chapters on cosmology.

Concentric Shells Thought Experiment: The concept that gravity increases the volume of the universe is important enough that another example will be given. Suppose that there are two concentric spherical shells around an origin point in space. The inside spherical shell is 4.4×10^9 m in circumference (about the circumference of the Sun). The outside shell is 2π meters larger circumference. With no mass at the origin and infinitely thin shells, this means that there is exactly a 1 meter gap between the shells. We could confirm the 1 meter spacing with a meter stick or a pulse of light and a clock.

Next we introduce the Sun's mass at the origin. This introduces gravity into the volume between the two shells with an average value of about $\Gamma \approx 1 + 2 \times 10^{-6}$. The circumference of each shell (proper length) does not change after we introduce gravity. However, the distance between the two shells would now be about $1 + 2 \times 10^{-6}$ meters. This is 2 microns larger than the zero gravity distance. This 2 micron increase in separation increases the proper volume between the two shells by roughly 10^{13} m^3.

In this example of two concentric shells we accepted the proper length of the circumference of the shells (tangential proper length). The question is whether there was also a decrease of this tangential length (relative to a “flat” coordinate system) when the sun’s mass was introduced at the origin. It is possible to consider both radial and tangential directions affected equally. This results in gravity producing an even larger increase in proper volume than previously calculated. This will be discussed further in the chapters on cosmology.

Connection Between the Rate of Time and Volume: We are going to compute the effect of the gravitational gamma Γ on proper volume using the standard Schwarzschild metric. The use of this metric means that the standard Schwarzschild conditions apply: a static nonrotating and uncharged spherically symmetric mass distribution in an empty universe. This calculation involves terminology from general relativity that is not explained here. Readers unfamiliar with general relativity should skip the shaded calculation section below and move on to the conclusion.

If we know metric equations, we can compute the 3-dimensional volume and the 4-dimensional volume (includes time). The easiest way to do it is to use the following diagonal metric:

$$dS^2 = -g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

Then the 3-dimensional volume dV(3) is:

$$dV(3) = (g_{11} \cdot g_{22} \cdot g_{33})^{1/2} dx^1 dx^2 dx^3$$

And for 4-dimensional volume $dV(4)$

$$dV(4) = (-g_{00} \cdot g_{11} \cdot g_{22} \cdot g_{33})^{1/2} dx^0 dx^1 dx^2 dx^3$$

In the particular case of the standard Schwarzschild metric:
$g_{00} = -1/\Gamma^2$; $g_{11} = \Gamma^2$; $g_{22} = R^2, g_{33} = R^2 sin^2\theta$

The differentials of 4 dimensional coordinates in this case are:
$(dx^0) = cdt$; $(dx^1) = dR$; $(dx^2) = d\theta$; $(dx^3) = d\Phi$ So:

$dV(3) = (\Gamma^2 \cdot R^2 \cdot R^2 sin^2\theta)^{1/2} dR \cdot d\theta \cdot d\Phi$
$dV(3) = \Gamma\ R^2 sin\theta\ dR\ d\theta\ d\Phi$ note that volume (3) scales with Γ

$V(4) = (-(-1/\Gamma^2) \cdot \Gamma^2 \cdot R^2 \cdot R^2 sin^2\theta)^{1/2} \cdot cdt \cdot dR \cdot d\theta \cdot d\Phi$
$dV(4) = R^2 csin\theta dt dR d\theta d\Phi$ note that this is independent of Γ

The above calculation shows that proper volume (3 spatial dimensions) scales with the gravitational gamma Γ. This supports the previous examples involving the volume increase interpretation of the Shapiro experiment and also the volume increase that occurs in the thought experiment with two concentric shells. When we include the time dimension and calculate the effect of the gravity generated by a single mass on the surrounding spacetime, we obtain the answer that the **4** dimensional spacetime volume is independent of gravitational gamma Γ. The radial dimension increases ($\Gamma = dL_R/dR$) and the temporal dimension decreases ($\Gamma = dt/d\tau$). These offset each other resulting in the 4 dimensional volume remaining constant.

There is a simpler way of expressing this concept. Since $\Gamma = \left(\frac{dt}{d\tau}\right) = \left(\frac{dL_R}{dR}\right)$ therefore:

$$d\tau dL_R = dtdR$$

This concept will be developed further when we develop particles out of 4 dimensional spacetime and it also has application to cosmology.